Enrollment No: ______________________ **Exam Seat No:**______________________

C.U.SHAH UNIVERSITY

Summer Examination-2016

Subject Name: Functional Analysis-I

Subject Code:5SCO2MTC3 **Branch:M.Sc.(Mathematics)**

Semester: 2 **Date :09/05/2016** **Time : 10:30 To 1:30** **Marks : 70**

Instructions:

(1) Use of Programmable calculator and any other electronic instrument is prohibited.
(2) Instructions written on main answer book are strictly to be obeyed.
(3) Draw neat diagrams and figures (if necessary) at right places.
(4) Assume suitable data if needed.

SECTION – I

Q-1 **Attempt the Following questions** **(07)**

a. Define: inner product space. **(01)**
b. State parallelogram law for inner product space. **(01)**
c. Define: orthonormal set **(01)**
d. State Schwarz's inequality. **(02)**
e. State Bessel's inequality. **(02)**

Q-2 **Attempt all questions** **(14)**

a) Let $<,>$ be an inner product on a linear space X and $T: X \to X$ be a linear be a linear one to one map. Let $< x, y >_T = < T(x), T(y) >, x, y \in X$ Then prove that $<,>_T$ is an inner product space. **(05)**

b) Let X be an inner product space. For $x \in X$, define $\|x\| = \sqrt{< x, x >}$. Then prove that $\| \quad \|$ is an norm on X. **(05)**

c) Let X be an inner product space and $\{x_n\}, \{y_n\}$ be sequence in X such that $\|x_n - x\| \to 0,\ \|y_n - y\| \to 0$ then prove that $< x_n, y_n > \to < x, y >$. **(04)**

OR

Q-2 **Attempt all questions** **(14)**

a) Let $X = K^n$, define $< x, y > = \sum_{j=1}^{n} x(j)\overline{y(j)}\ \forall x, y \in l^2$ then prove that $(K^n, <, >)$is an inner product space. **(05)**

b) State and prove Polarization identity for inner product space. **(05)**

c) State and prove Pythagoras theorem for inner product space. **(04)**

Q-3		**Attempt all questions**	**(14)**
	a)	Let X be an inner product space. Let $\{x_1, x_2, \ldots, x_n\}$ be a linearly independent subset of X. Define $y_1 = x_1$ with $u_1 = \frac{y_1}{\|y_1\|}$, for $n = 2,3, \ldots, y_n = x_n - < x_n, u_1 > u_1 - < x_n, u_2 > u_2 - \cdots \ldots \ldots - < x_n, u_{n-1} > u_{n-1}$ with $u_n = \frac{y_n}{\|y_n\|}$ then prove that $\{u_1, u_2, \ldots, u_n\}$ is an orthonormal set in X and for $n = 1,2, \ldots$ span $\{u_1, u_2, \ldots, u_n\}$ = span $\{x_1, x_2, \ldots, x_n\}$	**(06)**
	b)	If H be Hilbert space, $\{\alpha_n\}$ be a sequence in K and $\{u_1, u_2, \ldots\}$ be an orthonormal subset of H. Then prove that $\sum_{n=1}^{\infty} \alpha_n u_n$ converges in H if and only if $\sum_{n=1}^{\infty} \lvert\alpha_n\rvert^2 < \infty$.	**(04)**
	c)	Let X be an inner product space and E be an orthonormal subset of X. Then for each $x \in X$ prove that $E_x = \{u \in E : < x, u > \neq 0\}$ is countable.	**(04)**

OR

Q-3	**a)**	Let H be a Hilbert space and $E \subset H$ be orthonormal set. Then prove that following are equivalent (i) $E$is orthonormal basis. (ii) For each $x \in H, x = \sum_{n=1}^{\infty} < x, u_n > u_n$, where $E_x = \{u_1, u_2, \ldots.\}$ (iii)$L(E)$ is dense in H. (iv)For $x \in H$ and x is perpendicular to each element of E then $x = 0$.	**(06)**
	b)	Let H be a finite dimensional Hilbert space then prove that H is isometrically isomorphic to $(K^n, \| \ \|_2)$.	**(04)**
	c)	Let H be separable Hilbert space then prove that it has countable orthonormal basis.	**(04)**

SECTION – II

Q-4		**Attempt the Following questions**	**(07)**
	a.	Write an orthonormal basis for l^2.	**(02)**
	b.	Let H be a Hilbert space and $A \in BL(H)$ then prove that $\lambda \in \sigma(A)$ if and only if $\bar{\lambda} \in \sigma(A^*)$	**(02)**
	c.	If $A \in BL(H)$ is bounded below then $R(A^*) = H$. True or False.	**(01)**
	d.	Define: Self adjoint operator	**(01)**
	e.	Define: Numerical range of a bounded linear operator.	**(01)**

Q-5		**Attempt all questions**	**(14)**
	a)	Let H be a Hilbert space and Y be a closed subspace of H. Then prove that $H = Y \oplus Y^{\perp}$, where $Y^{\perp} = \{x \in H : < x, y > = 0, \forall\, y \in Y\}$ and $Y^{\perp\perp} = Y$. Also justify that completeness is essential in it.	**(08)**
	b)	State and prove unique Hahn – Banach extension theorem.	**(06)**

OR

Q-5	**a)**	State and prove Rietz – representation theorem for Hilbert space. Also justify that completeness is essential in it.	**(08)**
	b)	Let H be a Hilbert space. Define $j_y : H' \to K$ for $y \in H$ by $j_y(f) = f(y), f \in H'$ then prove that j_y is continuous linear functional on H' and $\|j_y\| = \|y\|$. Also if	**(06)**

$J: H \to H''$ by $J(y) = j_y$ then prove that J is an on to isometry isomorphism

Q-6 **Attempt all questions** **(14)**

a) Let A, B be two compact operators on H. Then prove that $A + B$, AC and CA are compact operators $\forall\, C \in BL(H)$. **(06)**

b) Let $K = C$ and $A \in BL(H)$. Then prove that there are unique self adjoint operators B and C on H such that $A = B + iC$. Furhther A is normal if and only if $BC = CB$. **(04)**

c) Let $A \in BL(H)$ be normal operator then prove that $\sigma(A) = \sigma_a(A)$ **(04)**

OR

Q-6 **Attempt all Questions**

a) Let H be a Hilbert space and $A \in BL(H)$ then prove that **(06)**

(i) $\sigma_e(A) \subset W(A)$.

(ii) $\sigma(A) \subset \overline{W(A)}$.

b) In usual notation prove that $\sigma_e(A) \subset \sigma_a(A) \subset \sigma(A)$. **(04)**

c) Let A and B be normal. If A commutes with B^* and B commutes with A^* then prove that $A + B$ and AB are normal. **(04)**